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Eulerian and Lagrangian renormalization in turbulence theory

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Systematic renormalized perturbation expansions for turbulence and turbulent convection are constructed which are invariant at each order under random Galilean transformations. Two types of expansion are developed whose lowest truncations give, respectively, the Lagrangian-history direct-interaction approximation and the abridged Lagrangian-history direct-interaction approximation. These approximations previously were derived as heuristic modifications of the Eulerian direct-interaction approximation (Kraichnan 1965). The techniques used involve reversion of primitive perturbation expansions for the generalized velocity field $\mathbf{u}(\mathbf{x}, t|s)$, defined as the velocity measured at time s in the fluid element which passes through $\mathbf{x}$ at time t. The new expansions are illustrated by application to a random linear oscillator, to passive-scalar convection by a random velocity and to the Lagrangian velocity covariance. The lowest term of the expansion for the passive scalar gives Taylor's (1921) exact result for dispersion of fluid elements, and higher terms describe the deviations of the particle-displacement distribution from Gaussian form. In all the applications the assumed underlying statistics are more general than Gaussian statistics, which appear as a special case.

1. Introduction

Renormalized perturbation expansions have been used in turbulence theory for some twenty years. Recently they have been given an elegant functional formulation by Martin, Siggia & Rose (1973) and Phythian (1975, 1976). These expansions, in which triple moments of the velocity field are expressed as infinite power series in the covariance of the velocity field, are markedly superior to primitive perturbation expansions in powers of the Reynolds number. Truncations of the latter give unphysical results except at very small Reynolds numbers while the simplest truncation of a renormalized expansion is the direct-interaction approximation (Kraichnan 1964*a*), which is self-consistent at all Reynolds numbers and in good numerical agreement with numerical simulations of isotropic turbulence at moderate Reynolds numbers (Orszag & Patterson 1972).

In one important respect, however, the renormalized expansions so far used are inferior to the primitive expansions. The former, but not the latter, give at every order a spurious effect of convection by large spatial scales upon energy transfer among small spatial scales (Kraichnan 1964*b*). This is because the renormalized expansions involve two-time Eulerian covariances and the correlation time of any Eulerian time-displaced average is controlled by the random displacement of velocity-field structures through convection by the large, energetic scales. Energy transfer, on the other hand, should be negligibly affected by large-scale convection if the large scales do not contribute significantly to the straining field. The infinite renormalized expansions must be summed completely to eliminate the spurious effects of convection on energy transfer. In the direct-interaction approximation, these effects give an incorrect inertial-range law.

In the present paper, we construct new renormalized expansions which are free of spurious convection effects at every order. In formal terms, each order of the new expansions is invariant under a random Galilean transformation (RGT). The RGT is defined as the addition to the turbulent velocity of a spatially uniform convecting velocity whose direction and amplitude vary randomly from one realization to another and whose statistical distribution may conveniently be taken as isotropic and Gaussian. Such a stochastic uniform velocity affects all Eulerian time-displaced averages but has no effect on energy transfer in homogeneous turbulence. The primitive perturbation expansions also are invariant under RGT at each order while the old renormalized expansions violate this invariance at each order.

The new expansions are constructed by breaking out of the Eulerian framework and dealing with a generalized velocity field $\mathbf{u}(\mathbf{x},t|s)$ defined as the velocity measured at time s in the fluid element whose space-time trajectory passes through $(\mathbf{x},t)$. Thus $\mathbf{u}(\mathbf{x},t|t)$ is just the Eulerian velocity $\mathbf{u}(\mathbf{x},t)$ while $\mathbf{u}(\mathbf{x},0|t)$ is the usual Lagrangian velocity if $t=0$ is the initial time. The generalized field obeys an equation like that of a passively advected field (Kraichnan 1965):

$$\partial\mathbf{u}(\mathbf{x},t|s)/\partial t = -\mathbf{u}(\mathbf{x},t).\nabla\mathbf{u}(\mathbf{x},t|s). \tag{1.1}$$

The RGT-invariant renormalized expansions are formed by a straightforward technique of series reversion which is easily described, although somewhat complicated in detail. There is no appeal to Feynman-type diagrams and the initial distribution of the turbulent velocity field need not be normal. The same technique gives an elementary derivation of the old, or Eulerian, renormalized expansions. The lowest truncations of two variants of the RGT-invariant renormalized expansions give respectively the Lagrangian-history direct-interaction (LHDI) and abridged Lagrangian-history direct-interaction (ALHDI) approximations, which previously (Kraichnan 1965) were derived as heuristic modifications of the Eulerian direct-interaction approximation. Now these approximations appear as the first steps in systematic expansions which are built from the primitive perturbation expansions without any use of the Eulerian renormalized expansions as intermediaries.

In order to give a brief qualitative explanation of the analysis to follow, we suppose that the distribution over the ensemble of the initial Eulerian velocity $\mathbf{u}(\mathbf{x},0)$ is homogeneous, isotropic and the sum of one or more multivariate-normal distributions which are identical except for the normalization of the total kinetic energy. Then all odd-order moments of $\mathbf{u}(\mathbf{x},0)$ vanish and all even-order moments are reducible to products of the velocity covariance. More general initial distributions can be handled, but they would add complications which are pointless for our present purpose. Let $\mathbf{u}^0(\mathbf{x},t)$ be the solution of the linearized Navier–Stokes equation (nonlinear terms removed), $G^0_{ij}(\mathbf{x},t;\mathbf{x}',t')$ be the Green's tensor of the linearized equation, and define the covariances

$$U_{ij}(\mathbf{x},t;\mathbf{x}',t') = \langle u_i(\mathbf{x},t)\,u_j(\mathbf{x}',t')\rangle, \quad U^0_{ij}(\mathbf{x},t;\mathbf{x}',t') = \langle u^0_i(\mathbf{x},t)\,u^0_j(\mathbf{x}',t')\rangle. \tag{1.2}$$

Finally let $G_{ij}(\mathbf{x}, t; \mathbf{x}', t')$ be the average Green's tensor for response of the actual field $\mathbf{u}(\mathbf{x}, t)$ to infinitesimal perturbations. Since $\mathbf{u}^0(\mathbf{x}, t)$ decays linearly from $\mathbf{u}(\mathbf{x}, 0)$, any moment of $\mathbf{u}^0$ can be expressed in terms of the covariance U^0.

If the nonlinear terms in the Navier–Stokes equation are reintroduced as a perturbation, an iteration process gives $\mathbf{u}(\mathbf{x}, t)$ as a functional power series in $\mathbf{u}^0$ and G^0. Any moment of $\mathbf{u}$ can then be developed as a power series in U^0 and G^0 by expanding each $\mathbf{u}$ factor in the moment, multiplying out the series, averaging, and reducing the averages to sums of products of covariances. A closely related procedure gives such a development for G also. These series in U^0 and G^0 are the primitive perturbation expansions.

The Eulerian renormalized expansion may now be constructed by the following steps.

(i) Revert the developments for U and G by iteration to yield expansions for G^0 and U^0 in functional powers of G and U.

(ii) Substitute the latter expansions for each G^0 and U^0 factor in the primitive expansion for the triple moments.

(iii) Multiply out and collect terms. The result is what is called the line-renormalized expansion for the triple moments, in which there appear powers of G and U only. In the special case where the initial distribution is simply normal, this renormalization accomplishes the elimination of self-energy parts, in the language of field theory. In the more general case we have taken, self-energy parts are not eliminated. A further reversion, whose description we defer, gives what is called the vertex-renormalized expansion for the triple moments.

The linearization of (1.1), with the right-hand side set equal to zero, gives simply

$$\mathbf{u}^0(\mathbf{x}, t|s) = \mathbf{u}^0(\mathbf{x}, s). \tag{1.3}$$

If the right-hand side of (1.1) is reintroduced as a perturbation, and (1.1), together with the Navier–Stokes equation, is solved iteratively, then the same steps as for the Eulerian field yield expansions for both the covariance $U_{ij}(\mathbf{x}, t|s; \mathbf{x}', t'|s')$ and the average infinitesimal Green's tensor $G_{ij}(\mathbf{x}, t|s; \mathbf{x}', t'|s')$ of the generalized field in powers of $U^0_{ij}(\mathbf{x}, t|s; \mathbf{x}', t'|s')$ and $G^0_{ij}(\mathbf{x}, t|s; \mathbf{x}', t'|s')$. In consequence of (1.3), the latter satisfy

$$U^0_{ij}(\mathbf{x}, t|s; \mathbf{x}', t'|s') = U^0_{ij}(\mathbf{x}, s; \mathbf{x}', s'), \quad G^0_{ij}(\mathbf{x}, t|s; \mathbf{x}', t'|s') = G^0_{ij}(\mathbf{x}, s; \mathbf{x}', s'). \tag{1.4}$$

The crucial step in forming the RGT-invariant renormalized expansions is a new reversion of the series for the generalized U and G in powers of U^0 and G^0. Since U and G depend non-trivially on four time arguments while, by (1.4), G^0 and U^0 depend only on two there is clearly more than one way to express the latter in terms of the former. The Eulerian reversion of step (i) therefore is not unique. An alternative reversion can be constructed by the following new steps:

(iv) In the primitive expansions for $U_{ij}(\mathbf{x}, t|s; \mathbf{x}', t|s')$ and $G_{ij}(\mathbf{x}, t|s; \mathbf{x}', t|s')$, the functions with labelling times t and t' equal, change the labelling times (preceding the vertical bars) in every U^0 and G^0 factor to t. By (1.4) this leaves every term unchanged in value.

(v) Revert by iteration to give expansions of $U^0_{ij}(\mathbf{x}, t|s; \mathbf{x}', t|s')$ and $G^0_{ij}(\mathbf{x}, t|s; \mathbf{x}', t|s')$ in functional powers of $U_{ij}(\mathbf{x}, t|s; \mathbf{x}', t|s')$ and $G_{ij}(\mathbf{x}, t|s; \mathbf{x}', t|s')$.

(vi) In the primitive expansions for triple moments with simultaneous labelling times t change all labelling times of G^0 and U^0 factors to t, again by (1.4).

(vii) Use the new expansions for U^0 and G^0 in powers of U and G to re-express the primitive expansions for the triple moments as expansions in powers of U and G, in analogy with the previous steps (ii) and (iii).

Since $U_{ij}(\mathbf{x},t|s;\mathbf{x}',t|s')$ and $G_{ij}(\mathbf{x},t|s;\mathbf{x}',t|s')$ trace events along simultaneously labelled fluid-element trajectories, they are RGT-invariant and so, therefore, are the new renormalized expansions for the simultaneously labelled triple moments.† The lowest-order truncation of these expansions gives the LHDI approximation. An elaboration of the reversion process extends it to triple moments with non-simultaneous labelling times and yields the full LHDI approximation of Kraichnan (1965).

If no truncation is made the Lagrangian-history (LH) expansion outlined above yields closed, infinite-series integro-differential equations for G and U with general time arguments $(t|s;t'|s')$. A closed subset of these equations involves only U and G with arguments of the form $(t|s;t|s')$, the purely Lagrangian functions. This is the counterpart of the closed, infinite-series equations for the purely Eulerian functions, with arguments of the form $(t|t;t'|t')$, obtained from the Eulerian renormalized expansion.

A second RGT-invariant renormalized expansion involves only the subset of Lagrangian functions with arguments of the form $(t|t;t|s)$, $t \geqslant s$. It is formed as follows (we suppress space arguments):

(viii) In the primitive expansions for $G(t|t;t|s)$ and $U(t|t;t|s)$ change every factor $G^0(t_1|s_1;t_2|s_2)$ and $U^0(t_1|s_1;t_2|s_2)$, $s_1 \geqslant s_2$, to $G^0(s_1|s_1;s_1|s_2)$ and $U^0(s_1|s_1;s_1|s_2)$, by authority of (1.4).

(ix) Revert to give $G^0(t|t;t|s)$ and $U^0(t|t;t|s)$ as functional power series in $G(t|t;t|s)$ and $U(t|t;t|s)$.

(x) Substitute these expansions into triple moments, again in analogy with steps (ii) and (iii).

The resulting abridged Lagrangian-history (ALH) renormalized expansion gives closed, infinite-series equations for $G(t|t;t|s)$ and $U(t|t;t|s)$. The lowest-order truncation is the ALHDI approximation.

In order to keep the present paper as little complicated as possible, the procedures outlined above are applied not to Navier–Stokes turbulence but to the analytically simpler problem of a passive scalar convected by a random velocity field which is constant in time in the Eulerian frame. The Eulerian renormalized series for this problem, and the reversion technique for generating them, have been discussed previously (Kraichnan 1970*a*) and there exist computer simulations of the scalar diffusion (Kraichnan 1970*b*, 1977). The LHDI approximation for the passive-scalar dynamics has also been presented before (Kraichnan 1965).

The analysis of the passive-scalar equations is preceded by that of a random linear oscillator, which is seen to be equivalent to the scalar problem in the degenerate case where the velocity field is spatially uniform. This permits the introduction of the methods in the simplest possible context. However, we retain the generalization to non-normal statistics because it makes clear some important features of the renormalized expansions.

† The total velocity field under the RGT is $\mathbf{u}+\mathbf{v}$, where $\mathbf{v}$ is the uniform field.

2. Renormalized expansions for the random oscillator

Let the scalar field $\psi(\mathbf{x},t)$ obey

$$(\partial/\partial t-\kappa\nabla^2)\,\psi(\mathbf{x},t) = -\mathbf{u}(\mathbf{x},t)\,.\,\nabla\psi(\mathbf{x},t),\quad \nabla\,.\,\mathbf{u}(\mathbf{x},t)=0, \tag{2.1}$$

where κ is a constant diffusivity. The generalized field $\psi(\mathbf{x},t|s)$, defined as the scalar amplitude at time s in the fluid element whose trajectory passes through $(\mathbf{x},t)$, then satisfies

$$\partial\psi(\mathbf{x},t|s)/\partial t = -\mathbf{u}(\mathbf{x},t)\,.\,\nabla\psi(\mathbf{x},t|s),\quad \psi(\mathbf{x},s|s)=\psi(\mathbf{x},s). \tag{2.2}$$

These equations may be Fourier-transformed with respect to $\mathbf{x}$ (Kraichnan 1965). Consider for now the degenerate case where $\mathbf{u}(\mathbf{x},t)=\mathbf{v}$, an isotropically distributed, uniform, time-independent velocity field. Clearly the different wave vectors uncouple and each wave-vector amplitude obeys equations of the form

$$(d/dt+\nu)\,z(t) = iaz(t), \tag{2.3}$$

$$\partial z(t|s)/\partial t = iaz(t|s),\quad z(s|s)=z(s), \tag{2.4}$$

where $\nu=\kappa k^2$, $a=-\mathbf{v}\,.\,\mathbf{k}$ and z is the amplitude for a wave vector $\mathbf{k}$.

The entire apparatus of renormalized perturbation expansions outlined in §1 is most simply illustrated by applying it to the generalized random oscillator defined by (2.3) and (2.4). We shall assume that odd moments of a vanish and denote the values of even moments by

$$M_{2n} = \langle a^{2n}\rangle. \tag{2.5}$$

Consider first the Eulerian Green's function $\hat{G}(t,t')=\hat{G}(t-t')$, defined as the solution of (2.3) if $z(t<0)=0$ and a forcing term $\delta(t-t')$ is added to the right-hand side. Thus

$$(d/dt+\nu)\,\hat{G}(t) = ia\hat{G}(t)\quad (t\geqslant 0),\quad \hat{G}(0)=1,\quad \hat{G}(t<0)=0. \tag{2.6}$$

This is equivalent to the integral equation

$$\hat{G}(t) = G^0(t)+ia\int_0^t G^0(t-s)\,\hat{G}(s)\,ds\quad (t\geqslant 0), \tag{2.7}$$

where $G^0(t)=\exp(-\nu t)$. The solution of (2.7) by iteration is

$$\hat{G}(t) = G^0(t)+iaG^0*G^0+(ia)^2G^0*G^0*G^0+\dots, \tag{2.8}$$

where $*$ denotes convolution and the argument t is implicit. The closed-form solution is of course $\hat{G}(t)=\exp(-\nu t+iat)$. The average of (2.8) over the distribution of a gives for $G(t)=\langle\hat{G}(t)\rangle$ the primitive perturbation series

$$G(t) = G^0(t)-M_2G^0*G^0*G^0+M_4G^0*G^0*G^0*G^0*G^0-M_6G^0(*G^0)^6+\dots. \tag{2.9}$$

Step (i) in the programme of §1 is the reversion of the functional power series (2.9). A general iterative method for such reversions is given in the appendix. Its application to (2.9) gives

$$G^0(t) = G(t)+M_2G*G*G+(3M_2^2-M_4)\,G(*G)^4+(12M_2^3-8M_2M_4+M_6)\,G(*G)^6+\dots. \tag{2.10}$$

Now substitute (2.8) into the right-hand side of (2.6), average and obtain the primitive perturbation expansion

$$\langle ia\hat{G}(t)\rangle = -M_2G^0*G^0+M_4G^0*G^0*G^0*G^0-M_6G^0(*G^0)^5+\dots. \tag{2.11}$$

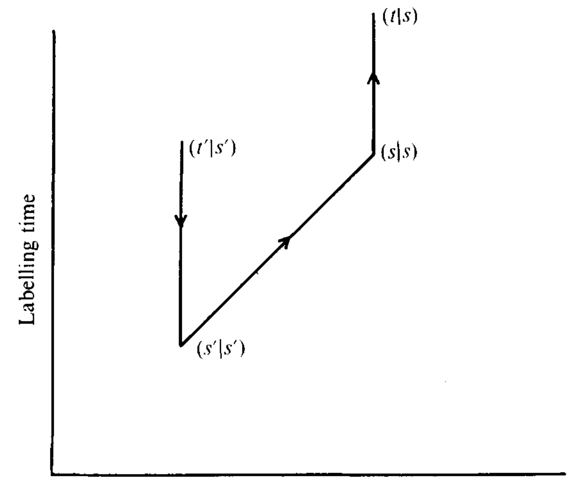

FIGURE 1. Integration path for (2.15). The values of t and t' are unrestricted, but $s \geqslant s'$ always.

Following steps (ii) and (iii) of § 1, substitute (2.10) for each G^0 factor in (2.11), multiply out and collect terms. The result is the Eulerian renormalized expansion

$$\langle ia\hat{G}(t)\rangle = -M_2 G*G + (M_4 - 2M_2^2)\, G(*G)^3 - (M_6 - 6M_2M_4 + 7M_2^3)\, G(*G)^5 + \ldots. \quad (2.12)$$

Equation (2.12), together with

$$(d/dt + \nu)\, G(t) = \langle ia\hat{G}(t)\rangle, \quad (2.13)$$

the average of (2.6), gives a closed, infinite-series integro-differential equation for G. If only the $M_2 G*G$ term in (2.12) is retained, the result is the direct-interaction approximation for G. The primitive and renormalized expansions have substantially different properties. With the integrations performed, (2.11) gives

$$\langle ia\hat{G}(t)\rangle = \exp(-\nu t)\,(-M_2 t + M_4 t^3/3! - M_6 t^5/5! + \ldots), \quad (2.14)$$

while, if the distribution of a is smooth, each term in (2.12) vanishes as $t \to \infty$ after the $\exp(-\nu t)$ has been factorized out of the convolutions.

Before discussing the renormalized expansions for the covariance of $z(t)$, we shall carry out the LH and ALH renormalizations for the averaged Green's function $G(t|s;t'|s')$ of the coupled equations (2.3) and (2.4). G is the average of $\hat{G}(t|s;t'|s')$, defined as the solution of (2.3) and (2.4) if $z(s < s') = 0$ and a forcing term $\delta(t-t')$ is added to the right-hand side of (2.4) at $s = s'$. Thus, as shown in figure 1, $\hat{G}$ is found by integrating (2.4) from t' to s', integrating (2.3) from s' to s, and finally integrating (2.4) from s to t. This corresponds to the integral equation

$$\begin{aligned}\hat{G}(t|s;t'|s') = G^0(t|s;t'|s') &+ ia\int_{t'}^{s'} G^0(t|s;s''|s')\,\hat{G}(s''|s';t'|s')\,ds'' \\ &+ ia\int_{s'}^{s} G^0(t|s;s''|s'')\,\hat{G}(s''|s'';t'|s')\,ds'' \\ &+ ia\int_{s}^{t} G^0(t|s;s''|s)\,\hat{G}(s''|s;t'|s')\,ds'' \quad (s \geqslant s'),\end{aligned} \quad (2.15)$$

where
$$G^0(t|s;t'|s') = \exp[-\nu(s-s')] \tag{2.16}$$
is the Green's function of the linearized ($a = 0$) equations. Inspection of (2.3) and (2.4) shows that the exact solution of (2.15) is
$$\hat{G}(t|s;t'|s') = \exp[ia(t-t')-\nu(s-s')] \quad (s \geqslant s'). \tag{2.17}$$
No inequalities restrict t and t' in (2.15)–(2.17). Equation (2.4) defines a labelling transformation which can go either forwards or backwards in time.

By (2.16) and (2.17), $G(t|s;t|s') = G^0(t|s;t|s')$, so that the ALH renormalization (steps (viii)–(x) of §1) is empty for G. This is not true for the full passive-scalar problem with non-uniform velocity fields. The analysis for the LH expansion is made clearer and more compact by using a shorthand notation for the time arguments and integrations. We shall denote $(t_1|t_2;t_3|t_4)$ by (1234) and write
$$\int_{t_3}^{t_1} dt_2 \ldots \int_{t_6}^{t_4} dt_5 = \int \left| \begin{matrix} 1 & & 4 \\ 2 & \ldots & 5 \\ 3 & & 6 \end{matrix} \right|.$$
In this notation (2.15) becomes
$$\hat{G}(1234) = G^0(1234) + ia \int \left| \begin{matrix} 4 \\ 5 \\ 3 \end{matrix} \right| G^0(1254)\,\hat{G}(5434) + ia \int \left| \begin{matrix} 2 \\ 5 \\ 4 \end{matrix} \right| G^0(1255)\,\hat{G}(5534)$$
$$+ ia \int \left| \begin{matrix} 1 \\ 5 \\ 2 \end{matrix} \right| G^0(1252)\,\hat{G}(5234) \quad (t_2 \geqslant t_4). \tag{2.18}$$

The primitive expansion for $G(1234)$ is formed by solving (2.18) iteratively and averaging. Thus (2.18) (with changed arguments) gives
$$\hat{G}(5434) = G^0(5434) + ia \int \left| \begin{matrix} 5 \\ 6 \\ 3 \end{matrix} \right| G^0(5464)\,\hat{G}(6434), \tag{2.19a}$$
$$\hat{G}(5534) = G^0(5534) + ia \int \left| \begin{matrix} 4 \\ 6 \\ 3 \end{matrix} \right| G^0(5564)\,\hat{G}(6434) + ia \int \left| \begin{matrix} 5 \\ 6 \\ 4 \end{matrix} \right| G^0(5566)\,\hat{G}(6634), \tag{2.19b}$$
$$\hat{G}(5234) = G^0(5234) + ia \int \left| \begin{matrix} 4 \\ 6 \\ 3 \end{matrix} \right| G^0(5264)\,G(6434) + ia \int \left| \begin{matrix} 2 \\ 6 \\ 4 \end{matrix} \right| G^0(5266)\,\hat{G}(6634)$$
$$+ ia \int \left| \begin{matrix} 5 \\ 6 \\ 2 \end{matrix} \right| G^0(5262)\,\hat{G}(6234), \tag{2.19c}$$
where cancellations of integration ranges due to equalities of arguments have been noted. To obtain the primitive expansion to third order in G^0, (2.19) is substituted into (2.18), each $\hat{G}$ is changed to G^0 and the result is averaged over a. To carry the

expansion to higher orders, each $\hat{G}$ in (2.19) is re-expressed using (2.18). The primitive expansion with all third-order terms shown explicitly is

$$
\begin{aligned}
G(1234) = {} & G^0(1234) - M_2 \int \left| \begin{matrix} 45 \\ 56 \\ 33 \end{matrix} \right| G^0(1254)\, G^0(5464)\, G^0(6434) \\
& - M_2 \int \left| \begin{matrix} 24 \\ 56 \\ 43 \end{matrix} \right| G^0(1255)\, G^0(5564)\, G^0(6434) - M_2 \int \left| \begin{matrix} 25 \\ 56 \\ 44 \end{matrix} \right| G^0(1255)\, G^0(5566)\, G^0(6634) \\
& - M_2 \int \left| \begin{matrix} 14 \\ 56 \\ 23 \end{matrix} \right| G^0(1252)\, G^0(5264)\, G^0(6434) - M_2 \int \left| \begin{matrix} 12 \\ 56 \\ 24 \end{matrix} \right| G^0(1252)\, G^0(5266)\, G^0(6634) \\
& - M_2 \int \left| \begin{matrix} 15 \\ 56 \\ 22 \end{matrix} \right| G^0(1252)\, G^0(5262)\, G^0(6234) + \text{higher-order terms.}
\end{aligned} \tag{2.20}
$$

The LH reversion of (2.20) requires that the labelling times be altered first, in accord with step (iv) of §1. By (2.16) all labelling times on the right-hand side of (2.20) may be altered freely without changing values. The LH alteration prescription is to change all intermediate labelling times (those that are integrated over) to t_1. The labelling time t_3 is unaltered. This is just step (iv) as stated in §1 if $t_1 = t_3$ and extends that step to $t_1 \neq t_3$. With the labelling times changed, reversion according to the appendix is accomplished up to the third order shown explicitly in (2.20) by simply exchanging the symbols G and G^0 and changing the signs of all the terms in M_2. At higher orders, the reversion is more complicated but proceeds straightforwardly. The result is then [step (v)]

$$
\begin{aligned}
G^0(1234) = {} & G(1234) + M_2 \int \left| \begin{matrix} 45 \\ 56 \\ 33 \end{matrix} \right| G(1214)\, G(1414)\, G(1434) \\
& + M_2 \int \left| \begin{matrix} 24 \\ 56 \\ 43 \end{matrix} \right| G(1215)\, G(1514)\, G(1434) + M_2 \int \left| \begin{matrix} 25 \\ 56 \\ 44 \end{matrix} \right| G(1215)\, G(1516)\, G(1634) \\
& + M_2 \int \left| \begin{matrix} 14 \\ 56 \\ 23 \end{matrix} \right| G(1212)\, G(1214)\, G(1434) + M_2 \int \left| \begin{matrix} 12 \\ 56 \\ 24 \end{matrix} \right| G(1212)\, G(1216)\, G(1634) \\
& + M_2 \int \left| \begin{matrix} 15 \\ 56 \\ 22 \end{matrix} \right| G(1212)\, G(1212)\, G(1234) + \text{higher-order terms.}
\end{aligned} \tag{2.21}
$$

This expansion may be simplified by noting that $G(1212) = G(1414) = 1$, by definition. Moreover, the integrations over measuring times (the times following the vertical bars in the uncompacted notation) are all simply convolutions, where they are not completely trivial. Thus the exponentials in ν which appear in (2.17) combine in (2.21) to give a simple factor $\exp[-\nu(t_2 - t_4)]$ in every term. The whole equation is identical with that for the case $\nu = 0$ except for multiplication by that factor. But for $\nu = 0$ there is no measuring-time dependence and each $G(mnrs)$ factor in (2.21) can be replaced without change of value by the corresponding Eulerian function $G(mmrr) = G(t_m - t_r)$. The κ dependence in the full passive-scalar problem does not factorize out in this simple way.

Equations (2.3) and (2.4) give for G the equations of motion

$$(\partial/\partial t_1 + \nu)\, G(1134) = \langle ia\hat{G}(1134)\rangle, \tag{2.22}$$

$$\partial G(1234)/\partial t_1 = \langle ia\hat{G}(1234)\rangle, \quad G(3434) = 1, \tag{2.23}$$

which are the differential equivalent of (2.15) averaged. The moment $\langle ia\hat{G}(1234)\rangle$ plays the same role for G as the triple moments discussed in § 1 do for the covariance. We generate the primitive expansion for $\langle ia\hat{G}(1234)\rangle$ by substituting (2.18), iterated to whatever order is desired explicitly, for $\hat{G}(1234)$ and averaging. The simplest case is $\langle ia\hat{G}(1222)\rangle$ because only the third integral in (2.18) survives. This case serves both to illustrate the methods and to display the key features of the LH expansion for the present model problem. The iteration gives

$$\begin{aligned}\langle ia\hat{G}(1222)\rangle = &-M_2\int\left|\begin{matrix}1\\3\\2\end{matrix}\right| G^0(1232)\, G^0(3222)\\ &+M_4\int\left|\begin{matrix}134\\345\\222\end{matrix}\right| G^0(1232)\, G^0(3242)\, G^0(4252)\, G^0(5222) - M_6\int\Big|\ \Big| \ldots.\end{aligned} \tag{2.24}$$

Step (vi) of § 1, the alteration of labelling times, changes this to

$$\begin{aligned}\langle ia\hat{G}(1222)\rangle = &-M_2\int\left|\begin{matrix}1\\3\\2\end{matrix}\right| G^0(1212)\, G^0(1222)\\ &+M_4\int\left|\begin{matrix}134\\345\\222\end{matrix}\right| G^0(1212)\, G^0(1212)\, G^0(1212)\, G^0(1222) - M_6\int\Big|\ \Big| \ldots.\end{aligned} \tag{2.25}$$

Finally, we generate the LH expansion by carrying out step (vii): we substitute (2.21), with properly named arguments, for every G^0 factor in (2.25).

Only $G^0(1212)$ and $G^0(1222)$ occur in (2.25), whatever the order. $G^0(1212)$ has the trivial reversion $G^0(1212) = G(1212)$; if $t_3 = t_1$ and $t_4 = t_2$, the integrals in (2.21) cancel identically at each order. For $G^0(1222)$, (2.21) gives

$$G^0(1222) = G(1222) + M_2\int\left|\begin{matrix}13\\34\\22\end{matrix}\right| [G(1212)]^2\, G(1222) + \text{higher-order terms.} \tag{2.26}$$

The LH expansion is now obtained by writing each $G^0(1212)$ factor in (2.25) as $G(1212)$ and substituting (2.26) for each $G^0(1222)$ factor.

The explicit form of the final result can be obtained readily, without doing all the details of substitution and collection of terms. In both (2.25) and (2.26) only the arguments (1212) and (1222) occur on the right-hand sides, and (1222) occurs only once in each term. This is a direct consequence of the prescription for changing labelling times and of the fact that only the third integral in (2.18) is involved in the iterations. Consequently, each term in the final result for $\langle ia\hat{G}(1222)\rangle$ is an integral over intermediate times with an integrand that is independent of those times and consists of a number of factors $G(1212) = 1$ and a single factor $G(1222)$. The integrations are then trivial. Since there are only odd numbers of integrations in (2.25), and even numbers in (2.26), each final term involves an odd number of integrations which, when performed, yield an odd power of $t = t_1 - t_2$. The final result must then have the form

$$dG(1222)/dt = \langle ia\hat{G}(1222)\rangle = \left[\sum_{n=1}^{\infty} C_{2n}(-1)^n t^{2n-1}/(2n-1)!\right] G(1222), \qquad (2.27)$$

where the C_{2n} can be determined by actually doing the work, and the factors $(-1)^n$ and $1/(2n-1)!$ simply affect the value of C_{2n}.

What are the C_{2n}? By (2.17), $G(1222) = \langle \exp(iat)\rangle$, the characteristic function of the probability distribution of a. The series in (2.27) is then the logarithmic derivative of the characteristic function, and the C_{2n} are simply the cumulants of the distribution of a. The first few are

$$C_2 = M_2, \quad C_4 = M_4 - 3M_2^2, \quad C_6 = M_6 - 15M_2M_4 + 30M_2^3. \qquad (2.28)$$

We have checked the analysis by verifying (2.28) explicitly up to the C_6 term from the LH expansion.

In the general case the LH renormalized expansion gives

$$\langle ia\hat{G}(1324)\rangle = \left[\sum_{n=1}^{\infty} C_{2n}(-1)^n t^{2n-1}/(2n-1)!\right] G(1324), \quad t = t_1 - t_2. \qquad (2.29)$$

This can be seen as follows. The alterations of labelling times leaves, in each term, one G factor with labelling times t_1 and t_2 and all the other factors with labelling times t_1 and t_1. The only dependence of the integrands on the intermediate times then comes from the exponentials of ν which occur according to (2.17) and which involve only measuring times. Since measuring times are unaltered, the original convolution character of the integrations over these times remains, and the ν exponentials factorize out of the convolutions to give a factor $\exp[-\nu(t_3 - t_4)]$ in front of each term. The rest of the argument goes like that for $\langle ia\hat{G}(1222)\rangle$. Again, we have verified the result.

Before completing the programme for the random oscillator by giving the LH and ALH expansions for triple moments, we wish to compare the primitive expansion (2.14), the Eulerian renormalized expansion (2.12) and the LH renormalized expansion (2.29). Both the primitive and the LH expansion give infinite-series differential equations for the time derivative of G, while (2.12) gives an infinite-series integro-differential equation. The pure differential character of the first two expansions carries over to the passive-scalar problem if $\kappa = 0$, but not for $\kappa > 0$, the reason being the mode dependence of the damping.

No truncation of (2.14) gives a valid approximation for all t. On the other hand (2.29) reduces to its leading term if the a distribution is Gaussian. When the higher cumulants do not vanish, any truncation of (2.29) whose last term has $C_{2n}(-1)^n$ negative gives an approximate differential equation for G whose solution is well behaved for all t and vanishes at $t = \infty$.

Equation (2.12) reduces to its leading term if a has a semicircle distribution. If the distribution is Gaussian, all higher truncations of (2.12) give integro-differential equations for G whose solutions blow up (Kraichnan 1961). For the semicircle distribution

$$G(t) = \exp(-\nu t)J_1(2a_0 t)/a_0 t, \quad a_0^2 = M_2. \qquad (2.30)$$

In this case, or any where the sign of $G(t)$ oscillates, the logarithmic derivative of $G(t)$, i.e. the sum of the series in square brackets in (2.29), has singularities. Thus (2.12) and (2.29) behave quite differently. The LH expansion is more appropriate when a is Gaussian or has moments that rise with order more rapidly than for a

Gaussian distribution. It leads more easily to well-behaved approximations in general. But the updating of labelling times gives an increased stability that makes representation of oscillations in $G(t)$ impossible by truncations of (2.29).

The LH reversion procedure was invoked in order that random Galilean invariance should survive renormalization. It is an unlooked for bonus that the final LH expansion for $G(t)$ should have a simpler and more transparent relation to the primitive moment expansion than has the Eulerian expansion, and give generally better approximations as well. There is no problem of Galilean invariance with the Eulerian function $G(t)$. Augmenting the equations of motion by a labelling transformation like (2.4) is a powerful procedure in other examples of stochastic dynamics where the question of random Galilean invariance does not enter at all.

Now denote $z(t_1|t_2)$ by $z(12)$ and assume initial values $z(t_0) = z(t_0|t_0) = z(00)$ which are statistically independent of a and distributed such that $\langle[z(00)]^2\rangle = \langle[z^*(00)]^2\rangle$. Define the covariance

$$Z(1234) = Z(3412) = \langle z(12)\,z(34)\rangle. \tag{2.31}$$

Under our assumptions Z is real, and if $z = x+iy$, x and y are uncorrelated, so that $Z(1111) = \langle[x(t_1)]^2\rangle - \langle[y(t_1)]^2\rangle$. By (2.3) and (2.4), Z obeys

$$(\partial/\partial t + \nu)\,Z(1134) = \langle iaz(11)\,z(34)\rangle, \tag{2.32}$$

$$\partial Z(1234)/\partial t = \langle iaz(12)\,z(34)\rangle. \tag{2.33}$$

If $\nu = 0$, $[x(t_1)]^2 + [y(t_1)]^2$ is conserved. The exact value of Z is

$$Z(1234) = \langle \exp[-\nu(t_2+t_4) + ia(t_1+t_3)]\rangle\, Z(0000), \tag{2.34}$$

where we take $t_0 = 0$.

Equations (2.3) and (2.4) have the linearized solution

$$z^0(12) = \exp(-\nu t_2)\,z(00) \tag{2.35}$$

and yield the integral equation

$$z(12) = z^0(12) + ia\int\left|\begin{matrix}2\\3\\0\end{matrix}\right| G^0(1233)\,z(33) + ia\int\left|\begin{matrix}1\\3\\2\end{matrix}\right| G^0(1232)\,z(32). \tag{2.36}$$

This represents integration of (2.3) along the diagonal segment of figure 2 followed by integration of (2.4) along the vertical segment. Iteration of (2.36) gives an expansion of $z(12)$ in G^0 and z^0 functions, and the averaged product of the series for $z(12)$ and $z(34)$ is the primitive expansion for $Z(1234)$. Thus, for example,

$$\begin{aligned}
Z(1134) = {} & Z^0(1134) - M_2\int\left|\begin{matrix}14\\56\\00\end{matrix}\right| G^0(1155)\,\underline{G}^0(3466)\,Z^0(5566) \\
& - M_2\int\left|\begin{matrix}13\\56\\04\end{matrix}\right| G^0(1155)\,\underline{G}^0(3464)\,Z^0(5564) - M_2\int\left|\begin{matrix}15\\56\\00\end{matrix}\right| G^0(1155)\,G^0(5566)\,Z^0(6634) \\
& - M_2\left|\begin{matrix}45\\56\\00\end{matrix}\right| \underline{G}^0(3455)\,\underline{G}^0(5566)\,Z^0(1166) - M_2\int\left|\begin{matrix}34\\56\\40\end{matrix}\right| \underline{G}^0(3454)\,\underline{G}^0(5466)\,Z^0(1166) \\
& - M_2\int\left|\begin{matrix}35\\56\\44\end{matrix}\right| \underline{G}^0(3454)\,\underline{G}^0(5464)\,Z^0(1164) + \text{higher-order terms}.
\end{aligned} \tag{2.37}$$

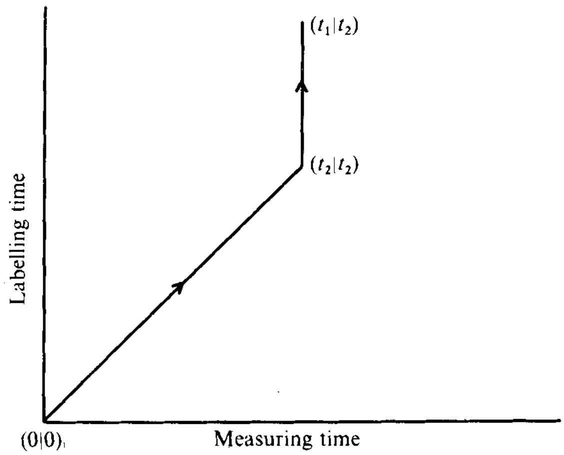

FIGURE 2. Integration path for (2.36). The argument t_1 may have any value $t_1 \geqslant 0$.

Here the underlined G^0 factors are the ones that appear in the expansion of $z(34)$ and the second pair of arguments in each Z^0 denotes a z^0 factor in that expansion. There are five more terms of second order in G^0 in the general case $Z(1234)$.

The Eulerian renormalized expansion for $\langle iaz(11)\,z(33)\rangle$ is obtained by repeating steps (i)–(iii) of § 1. Equation (2.37) is reverted, in two stages, to give an expansion of $Z^0(1133)$ in powers of the Eulerian Z and G. First, reversion by the method of the appendix gives a series in Z and G^0; then the G^0 factors are expressed in terms of G by (2.10). The result is used, together with (2.10), to express all the G^0 and Z^0 factors in the primitive expansion of $\langle iaz(11)\,z(33)\rangle$ as series in G and Z.

For the LH renormalized expansion of $\langle iaz(12)\,z(34)\rangle$, steps (iv)–(vii) are performed with the following elaboration of the relabelling prescription to the general case $t_1 \neq t_3$: in the primitive expansions for the moments $Z(1234)$ and $\langle iaz(12)\,z(34)\rangle$ change each intermediate labelling time in the expansion of the factor $z(12)$ to t_1 and each intermediate labelling time in the expansion of $z(34)$ to t_3. Thus in (2.37) the labelling times in every G^0 not underlined and in the first argument pair of every Z^0 are changed to t_1 while all other intermediate labelling times are changed to t_3. This prescription is that of Kraichnan (1965). It generalizes to products of any number of z factors. It is consistent with the prescription already used for the expansion of $G(1234)$ since $G(1234)$ is in fact the average of $z(12)$ with a particular initial condition.

We shall omit intermediate steps, which follow those for G detailed before, and give the final results. The complete lowest-order analysis for the full passive-scalar problem is given in Kraichnan (1965). The LH expansion gives a formula like (2.29):

$$\langle iaz(12)\,z(34)\rangle = \left[\sum_{n=1}^{\infty} C_{2n}(-1)^n t^{2n-1}/(2n-1)!\right] Z(1234), \quad t = t_1+t_3. \tag{2.38}$$

By (2.34) and (2.17) the closed form for $Z(1234)$ is

$$Z(1234) = G(t_1+t_3|t_2+t_4\,;\,0|0)\,Z(0000). \tag{2.39}$$

The lowest truncation of (2.38) is thus an exact expression for $\partial Z(1234)/\partial t_1$ if the distribution of a is Gaussian. The Eulerian expansion can be manipulated into the form

$$\langle iaz(11)\,z(33)\rangle = S[G(t)]\,Z(0000), \quad t = t_1+t_3, \tag{2.40}$$

where $S[G(t)]$ is the right-hand side of (2.12). Thus, again, the Eulerian expansion is exact at the first truncation if a has a semicircle distribution.

The ALH renormalization according to steps (viii)–(x) of §1 brings in some new features. We deal now only with arguments of the form (1114). As noted already, the ALH renormalization for the Green's function is empty since $G(1114) = G^0(1114)$. The ALH relabelling of the right-hand side of (2.37), with $t_3 = t_1$ and $t_1 \geq t_4$, is obtained by altering all labelling times, intermediate or not, in the Z^0 factors so that each $G^0(mnrs)$ is changed to $G^0(nnns)$ and each $Z^0(mnrs)$ is changed to $\tilde{Z}^0(ns)$, where

$$\tilde{Z}^0(ns) = Z^0(nnns) \quad (t_n \geq t_s), \quad \tilde{Z}^0(ns) = Z^0(snss) \quad (t_n < t_s). \tag{2.41}$$

Again, by (2.16) and (2.35) this changes no values. Equation (2.37) is then reverted and each G^0 is written as the G with the same arguments. The same alteration of labelling times is then made in the primitive expansion for $\langle iaz(11)\, z(14)\rangle$ and each G^0 and $\tilde{Z}^0$ in the altered expansion is replaced by G and by the reversion of (2.37) respectively. The result, after multiplying out and collecting terms, is the ALH renormalized expansion for $\langle iaz(11)\, z(14)\rangle$, in which there appear only functions of the form $G(nnns)$ and $\tilde{Z}(ns)$.

The algebra is simplest if $\nu = 0$, in which case there is no dependence of any function on measuring times and all the G functions which enter are equal to one. Nothing is lost by taking this case since, as before, the measuring-time dependence in the more general case is essentially trivial. If $\nu = 0$, the final expansion can be written as

$$\langle ia[z(t)]^2\rangle = -2h(t)\, Z(t) + \int_0^t f(t,s)\, Z(s)\, ds, \tag{2.42}$$

where
$$h(t) = M_2 t - (M_4 - 6M_2^2)\, t^3/3! + \text{higher-order terms}, \tag{2.43}$$

$$f(t,s) = M_4(4ts - 3s^2) - 4M_2^2(t^2 - ts) + \text{higher-order terms} \tag{2.44}$$

and $Z(t) = \langle [z(t)]^2\rangle$. Note that because of the measuring-time independence $z(14) = z(11)$, so that (2.42) is actually the most general triple moment in the ALH set for $\nu = 0$.

Equation (2.42) resembles both the LH and Eulerian expansions in that there are both completely updated terms, proportional to $Z(t)$, and integrals over the past history of Z. If the distribution of a is Gaussian, the leading term of (2.42) is exact $[h(t) = M_2 t, f(t,s) = 0]$. Truncation at this order is identical with the LH truncation. However, (2.42) has the peculiar feature that the remainder of the series does not vanish order by order for a Gaussian a, as does the LH expansion. Instead, if each order $2n$ $(n \geq 2)$ is expanded in powers of t, using the exact $Z(t)$, the leading contribution, proportional to t^{2n-1}, cancels and there is a residue of higher powers which are cancelled by the higher-order terms. Only the sum of all the orders $n \geq 2$ vanishes to all powers of t. Thus if a is Gaussian (2.42) is no longer exact if truncated at $2n > 2$.

3. Green's function for a passive-scalar field

Now we return to the full passive-scalar dynamics described by (2.1) and (2.2). Let $\mathbf{u}(\mathbf{x}, t)$ be a velocity field with prescribed isotropic and stationary statistics. In correspondence with §2, we wish to treat statistics more general than Gaussian in order to display better the structure of the expansions. For this purpose we take the distribution of $\mathbf{u}$ to be a sum of multivariate-normal distributions whose covariance functions differ only by a normalization factor. This has the consequence that moments of order $2n > 2$ factorize into sums of products of covariances, as in a normal distribution, but exceed normal values by ratios $R_{2n} \geqslant 1$, the values $R_{2n} = 1$ being reached only when the total distribution is strictly normal. Finally, we assume mirror-symmetric statistics.

We also admit an RGT with velocity $\mathbf{v}$ which convects both the scalar field and the random velocity field $\mathbf{u}(\mathbf{x}, t)$. Under this transformation, which is switched on at $t = 0$,

$$\mathbf{u}(\mathbf{x}, t) = [\mathbf{u}(\mathbf{x}-\mathbf{v}t, t)]_0, \quad \psi(\mathbf{x}, t|s) = [\psi(\mathbf{x}-\mathbf{v}t, t|s)]_0, \tag{3.1}$$

where the left-hand sides are values with the transformation and the right-hand sides those without. The total convecting velocity acting on ψ is $\mathbf{u}+\mathbf{v}$.

Let $G(\mathbf{x}, t|s; \mathbf{x}', t'|s')$ be the average Green's function for (2.1) and (2.2), and $G(\mathbf{x}, t; \mathbf{x}', t')$ the Eulerian Green's function for (2.1) alone. Then

$$G(\mathbf{x}, t; \mathbf{x}', t') = G(\mathbf{x}, t|t; \mathbf{x}', t'|t'), \quad G(\mathbf{x}, t'|s'; \mathbf{x}', t'|s') = \delta(\mathbf{x}-\mathbf{x}'). \tag{3.2}$$

$G(\mathbf{x}, t|s; \mathbf{x}', t'|s')$ corresponds to $G(t|s; t'|s')$ of §2, and the whole perturbation-expansion, relabelling and reversion apparatus of §2 carries over to the present problem with the sole change that intermediate integrations are over co-ordinates as well as times. In what follows we shall therefore omit most of the steps in the derivations. They can be filled in with §2 as a guide. The basic perturbation expansion is given in detail to lowest order by Kraichnan (1965) and the Eulerian expansion is given to fourth order by Kraichnan (1970*a*).

We set $\kappa = 0$. This has several advantages. G becomes the probability distribution for dispersion of single fluid elements by the motion. The analysis is greatly simplified because (2.1) and (2.2) are the same with the result that there is no dependence at all on measuring times:

$$G(\mathbf{x}, t|s; \mathbf{x}', t'|s') = G(\mathbf{x}, t; \mathbf{x}', t'), \quad t \geqslant t'.$$

Finally, the elimination of linear damping shows up more sharply the deficiencies of approximations. The demands on approximations are also more severe when the Eulerian velocity field is frozen, $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x})$, except for a possible RGT. We shall therefore be principally interested in that case, which gives additional simplifications. In the opposite extreme, very rapid time variation of $\mathbf{u}(\mathbf{x}, t)$, all the renormalized expansions reduce to their leading terms, which then are equivalent.

If we write

$$G(\mathbf{x}-\mathbf{x}', t-t') = G(\mathbf{x}, t; \mathbf{x}', t'), \quad U_{ij}(\mathbf{x}-\mathbf{x}', t-t') = U_{ij}(\mathbf{x}, t; \mathbf{x}', t'),$$

the Eulerian renormalized expansion for $\partial G(\mathbf{x}, t)/\partial t$ can be expressed as

$$\partial G(\mathbf{x}, t)/\partial t = \int_0^t ds \int d\mathbf{y}\, \eta(\mathbf{x}-\mathbf{y}, t-s)\, \nabla^2 G(\mathbf{y}, s)\, ds, \quad G(\mathbf{x}, 0) = \delta(\mathbf{x}), \tag{3.3}$$

where

$$\begin{aligned}\eta(\mathbf{x}, t) = {} & \tfrac{1}{3} U_{ii}(\mathbf{x}, t)\, G(\mathbf{x}, t) + \frac{1}{3}\int_0^t ds \int_0^s ds' \int d\mathbf{y} \int d\mathbf{y}' [(R_4-1)\, U_{ij}(\mathbf{x}-\mathbf{y}, t-s)\, U_{mi}(\mathbf{y}', s') \\ & + (R_4-1)\, U_{ii}(\mathbf{x}, t)\, U_{jm}(\mathbf{y}-\mathbf{y}', s-s') + R_4 U_{im}(\mathbf{x}-\mathbf{y}', t-s')\, U_{ji}(\mathbf{y}, s)] \\ & \times G(\mathbf{x}-\mathbf{y}, t-s)\, [\partial G(\mathbf{y}-\mathbf{y}', s-s')/\partial y_j]\, [\partial G(\mathbf{y}', s')/\partial y'_m] + \text{higher-order terms}.\end{aligned} \tag{3.4}$$

This corresponds to (2.12). The diffusivity acting on uniform scalar gradients is then

$$K(t) = \int_0^t ds \int d\mathbf{y}\, \eta(\mathbf{y}, s), \tag{3.5}$$

if the $\mathbf{u}$ field is switched on at $t = 0$.

The terms in (3.4) proportional to $R_4 - 1$ disappear when the $\mathbf{u}$ distribution is Gaussian ($R_4 = 1$). They are self-energy terms in the diagram language of field theory. Truncation of (3.4) at the term linear in U gives the direct-interaction approximation. The function $K(t)$ computed from this approximation agrees excellently with computer simulations of dispersion in the critical case of a frozen velocity field (Kraichnan 1970*b*) with Gaussian statistics. The direct-interaction approximation does less well in representing the short-time behaviour of the full function $G(\mathbf{x}, t)$. For t small enough that typical fluid elements have travelled much less than a correlation length, $\mathbf{u}$ is effectively a uniform field and, as noted after (2.3), the Fourier transform of $G(\mathbf{x}, t)$ is equivalent to the random oscillator $G(t)$. The direct-interaction approximation for the latter is exact for a semicircle distribution of a, and the consequence is that $\int G(\mathbf{x}, t)\, dx_2 dx_3$, the probability distribution for dispersion along the x_1 axis, is also a semicircle distribution in this approximation, while the full function $G(\mathbf{x}, t)$ for short t has an unphysical cusped shape (Roberts 1961). For times long compared with an eddy circulation time, the direct-interaction approximation for $G(\mathbf{x}, t)$ has a Gaussian shape.

Our primary concern is behaviour under an RGT. The exact $G(\mathbf{x}, t)$ changes under (3.1) according to the relation

$$G(\mathbf{x}, t) = \exp[\tfrac{1}{2}v_0^2 t^2 \nabla^2][G(\mathbf{x}, t)]_0 = (2\pi v_0^2 t^2)^{-\frac{3}{2}} \int \exp(-|\mathbf{x}-\mathbf{y}|^2/2v_0^2 t^2)[G(\mathbf{y}, t)]_0 d\mathbf{y}, \tag{3.6}$$

where v_0 is the root-mean-square value of any vector component of the Gaussianly distributed $\mathbf{v}$ field. The exact behaviour of $U(\mathbf{x}, t)$ is

$$U_{ij}(\mathbf{x}, t) = \exp(\tfrac{1}{2}v_0^2 t^2 \nabla^2)[U_{ij}(\mathbf{x}, t)]_0. \tag{3.7}$$

From (3.6) it follows that the total diffusivity with $\mathbf{v}$ switched on is

$$\frac{1}{6}\frac{d}{dt}\int x^2 G(\mathbf{x}, t)\, dx = v_0^2 t + \frac{1}{6}\frac{d}{dt}\int x^2 [G(\mathbf{x}, t)]_0 dx = v_0^2 t + [K(t)]_0, \tag{3.8}$$

where we use, twice, the commutation relation $\mathbf{x} f(\nabla) = f(\nabla)\mathbf{x} - \partial f(\nabla)/\partial \nabla$. Thus the diffusivities from the $\mathbf{v}$ field alone and from the $\mathbf{u}$ field alone are simply additive.

Equation (3.8) is violated at each order of (3.4). Consider the leading term. The total velocity covariance with $\mathbf{v}$ switched on is $\delta_{ij} v_0^2 + U_{ij}(\mathbf{x}, t)$ and by definition $\int G(\mathbf{x}, t)\, dx = 1$. The leading term thereby generates the $v_0^2 t$ term in (3.8). But by (3.6) and (3.7), $G(\mathbf{x}, t)$ and $U_{ij}(\mathbf{x}, t)$ are both smeared out in space by the RGT, so that the integral over space in (3.5) is diminished. The direct-interaction expression for $K(t)$ thus does not exhibit invariance under the RGT as does the exact $K(t)$. The same smearing phenomenon affects the values of each of the higher-order contributions to (3.4), and invariance is recovered only if the whole (divergent) series is summed.

This particular violation of RGT invariance is not a serious practical matter because the one-particle diffusivity is usually dominated by the large scales, so that an incorrect representation of the added contribution of small scales is unimportant. However the mechanism of violation is essentially the same as in the more important cases of transfer of scalar variance and the behaviour of the two-particle diffusivity, and also in the case of energy transfer in Navier–Stokes dynamics (Kraichnan 1964*b*).

The LH renormalized expansion for

$$\partial G(\mathbf{x}-\mathbf{x}', t-t')/\partial t = -\langle \mathbf{u}(\mathbf{x},t).\nabla \hat{G}(\mathbf{x},t;\mathbf{x}',t')\rangle \tag{3.9}$$

is constructed according to the programme in § 1 by changing all labelling times in the primitive expansion of the right-hand side and reverting to express U^0 and G^0 functions as series in U and G functions. Since we prescribe the **u** statistics we lack an equation of motion for $\mathbf{u}(\mathbf{x},t)$ to use in this process. In the frozen case $\mathbf{u}(\mathbf{x},t) = \mathbf{u}(\mathbf{x})$ we have simply

$$u^0(\mathbf{x},t|s) = \mathbf{u}(\mathbf{x}) \tag{3.10}$$

and the dynamics of $\mathbf{u}(\mathbf{x},t|s)$ are fully given by the labelling transformation

$$\partial \mathbf{u}(\mathbf{x},t|s)/\partial t = -\mathbf{u}(\mathbf{x}).\nabla\mathbf{u}(\mathbf{x},t|s), \quad \mathbf{u}(\mathbf{x},s|s) = \mathbf{u}(\mathbf{x}). \tag{3.11}$$

More generally, a wide variety of time dependencies for $U_{ij}(\mathbf{x},t;\mathbf{x}',t')$ can be realized by adjoining to (3.11) an artificial equation of motion for $\mathbf{u}(\mathbf{x},t)$ of the form

$$\partial \tilde{u}(\mathbf{k},t)/\partial t = ika(\mathbf{k},t)\,\tilde{\mathbf{u}}(\mathbf{k},t), \tag{3.12}$$

where $\tilde{\mathbf{u}}(\mathbf{k},t)$ is a wave-vector amplitude of $\mathbf{u}(\mathbf{x},t)$ and the $a(\mathbf{k},t)$ are stationary random processes, which may be independent for each pair $(\mathbf{k}, -\mathbf{k})$. A special case of (3.12) is

$$\partial \mathbf{u}(\mathbf{x},t)/\partial t = \mathbf{w}(t).\nabla\mathbf{u}(\mathbf{x},t), \tag{3.13}$$

where $\mathbf{w}(t)$ is a statistically stationary random uniform velocity field, a time-dependent RGT in other words. Our principal interest here is in the frozen case.

It is instructive to form the LH renormalization for $\partial G(\mathbf{x},t)/\partial t$ in two stages. In the first stage, reversions are carried out for the scalar field only, with $\mathbf{u}(\mathbf{x},t)$ treated as a given parameter field. In the second, the programme of § 1 is completed by doing relabellings and reversions on the **u** field also. The first stage is in precise correspondence with the analysis of § 2, with the simplification that since $\kappa = 0$ all the G functions are independent of measuring times.

After the change of labelling times in the first stage, only two types of argument sets appear in the G^0 and later G functions which enter the expansion of the right-hand side of (3.9), $(\mathbf{y},t|s;\mathbf{y}',t|s')$ and $(\mathbf{y},t|s;\mathbf{x}',t'|t')$. In each term, there is only one function with arguments of the second type, and of the other factors in the term, of the first type, there is one with $\mathbf{y} = \mathbf{x}$, $s = t$ [cf. the discussion preceding (2.27)]. But because of the measuring-time independence,

$$\left.\begin{aligned} G(\mathbf{y},t|s;\mathbf{y}',t|s') = G(\mathbf{y},t|t;\mathbf{y}',t|t) = \delta(\mathbf{y}-\mathbf{y}'),\\ G(\mathbf{y},t|s;\mathbf{x}',t'|t') = G(\mathbf{y},t|t;\mathbf{x}',t'|t') = G(\mathbf{y}-\mathbf{x}',t-t').\end{aligned}\right\} \tag{3.14}$$

Thus in each term of the final LH renormalized expansion all but one G factor is a δ-function. The space integrals therefore collapse and after they have been performed the remaining G factor becomes $G(\mathbf{x}-\mathbf{x}',t-t')$. The U factors, which all have argument **x** or intermediate space arguments before the integrations, collapse to single-point averages.

The series so obtained is

$$\partial G(\mathbf{x},t)/\partial t = \sum_{n=1}^{\infty} K_{2n}^{0}(t)\nabla^{2n}G(\mathbf{x},t), \quad G(\mathbf{x},0) = \delta(\mathbf{x}), \tag{3.15}$$

where

$$K_2^0(t) = \frac{1}{3}\int_0^t U(s)\,ds - \tfrac{1}{9}R_4\int_0^t ds\int_0^s ds'\int_0^{s'} ds''B(t-s'')\,U(s-s') + \text{higher-order terms},$$

$$K_4^0(t) = \tfrac{1}{9}(R_4-1)\int_0^t ds\int_0^t ds'\int_0^{s'} ds''\,U(t-s)\,U(s'-s'') + \text{higher-order terms} \tag{3.16}$$

and

$$U(t) = U_{ii}(0,t), \quad B(t-t') = -\langle u_i(\mathbf{x},t)\nabla^2 u_i(\mathbf{x},t')\rangle = \left\langle \frac{\partial u_i(\mathbf{x},t)}{\partial x_j}\,\frac{\partial u_i(\mathbf{x},t')}{\partial x_j}\right\rangle.$$

The leading term of each $K_{2n}^0(t)$ is of degree n in U. The higher-order contributions to all the $K_{2n}^0(t)$ include terms with averages over space derivatives of $\mathbf{u}$.

The first-stage renormalized expansion, an Eulerian–Lagrangian bastard, is markedly inferior to the straight Eulerian expansion (3.3), and its truncations give very bad approximations. $K_2^0(t)$ in (3.15) is a formally exact infinite-series representation of the eddy diffusivity $K(t)$. But truncation of $K_2^0(t)$ at its leading term is an unacceptable approximation. The G factor which appears in the leading term of (3.4) is missing, and $U(s)$ carries no information about the spatial structure of the velocity field. In the frozen-field case, where $U(s)$ is constant, truncation of $K_{2n}^0(t)$ at the leading term gives a diffusivity that increases linearly with t forever and corresponds exactly to that of a random, frozen, spatially uniform velocity field. Moreover, the violation of RGT invariance noted for truncations of (3.4) continues to afflict truncations of (3.16).

Note that (3.15) does not reduce to its leading term when $\mathbf{u}$ is Gaussian, so that $R_n = 1$. This contrast with (2.29), despite the correspondence of renormalization procedures in the two cases, is a consequence of the present system having many coupled degrees of freedom.

The second stage in constructing the LH renormalized expansion for $\partial G(\mathbf{x},t)/\partial t$ is to express all the U functions and spatial derivatives thereof in the expansions (3.16) for the $K_{2n}^0(t)$ in terms of U^0 functions, change labelling times in the U^0 functions according to the LH prescription, and then re-express the U^0 functions in terms of U functions with the altered labelling times. This is made simpler in the frozen-$\mathbf{u}$ case by the fact that the U functions in (3.16) are already the U^0 functions. We shall now carry out the steps for the frozen-field case up to second order, which is sufficient to write out the final expansion for $\partial G(\mathbf{x},t)/\partial t$ up to fourth order (terms quadratic in U).

We start by forming the primitive expansion of $\mathbf{u}(\mathbf{x},t|s)$ through iteration of the integral-equation equivalent to (3.11). Let G_L denote the Green's function for (3.11). The function G_L^0 for the linearized equation is simply

$$G_L^0(\mathbf{x},t|s;\mathbf{x}',t'|s) = \delta(\mathbf{x}-\mathbf{x}'). \tag{3.17}$$

Now we can restrict attention to averages with simultaneous labelling times t since only such averages appear in (3.16) and, as stated in § 1, the equations for such averages form a closed subset in the LH scheme. This means that after the change of labelling times in the primitive expansion only the function $G_L^0(\mathbf{x},t|s;\mathbf{x}',t|s)$, which has the trivial reversion

$$G_L^0(\mathbf{x},t|s;\mathbf{x}',t|s) = G_L(\mathbf{x},t|s;\mathbf{x}',t|s) = \delta(\mathbf{x}-\mathbf{x}'), \tag{3.18}$$

appears. Note that s is the only measuring time which enters G functions in the integration of (3.11); it is a constant parameter. This all means that we can forget Green's functions completely in handling (3.11) and instead simply iterate the integral of (3.11) with respect to t:

$$u_i(\mathbf{x},t|s) = u_i^0(\mathbf{x},t|s) - \int_s^t ds'\, u_j^0(\mathbf{x},s'|s')\,\nabla_j u_i^0(\mathbf{x},s'|s) + \int_s^t ds' \int_s^{s'} ds''\, u_m^0(\mathbf{x},s'|s')\,\nabla_m u_j^0(\mathbf{x},s''|s'')\,\nabla_j u_i^0(\mathbf{x},s''|s) + \ldots, \quad (3.19)$$

where $\nabla_j = \partial/\partial x_j$ and $u_i^0(\mathbf{x},t|s) = u_i^0(\mathbf{x},s|s) = u_i(\mathbf{x})$, the frozen field.

The averaged product of (3.19) for two values of $\mathbf{x}$ and s is

$$\begin{aligned} U_{in}(\mathbf{x},t|s_1;\mathbf{x}',t|s_2) = {} & U_{in}^0(\mathbf{x},t|s_1;\mathbf{x}',t|s_2) \\ & + R_4 \int_{s_1}^t ds' \int_{s_2}^t ds''\, U_{jm}^0(\mathbf{x},s'|s';\mathbf{x}',s''|s'')\,\nabla_j \nabla'_m U_{in}^0(\mathbf{x},s'|s_1;\mathbf{x}',s''|s_2) \\ & + R_4 \int_{s_1}^t ds' \int_{s_1}^{s'} ds''\, U_{mj}^0(\mathbf{x},s'|s';\mathbf{x},s''|s'')\,\nabla_j \nabla_m U_{in}^0(\mathbf{x},s''|s_1;\mathbf{x}',t|s_2) \\ & + R_4 \int_{s_2}^t ds' \int_{s_2}^{s'} ds''\, U_{mj}^0(\mathbf{x}',s'|s';\mathbf{x}',s''|s'')\,\nabla'_j \nabla'_m U_{in}^0(\mathbf{x},t|s_1;\mathbf{x}',s''|s_2) \\ & + \text{higher-order terms}, \end{aligned} \quad (3.20)$$

where $\nabla'_j = \partial/\partial x'_j$ and terms which vanish by incompressibility, mirror-symmetry or homogeneity have been omitted. The LH alteration prescription is to change all labelling times in (3.20) to t.

Before reverting (3.20), with altered labelling times, we must deal with a complication which was skipped over in §1. The fields $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{u}^0(\mathbf{x},t|s)$ are solenoidal. But in general $\mathbf{u}(\mathbf{x},t|s)$ is not because there is no pressure term in (3.11). Clearly every order of the reverted series for the solenoidal tensor $U_{ij}^0(\mathbf{x},t|s_1;\mathbf{x},t|s_2)$ should be solenoidal. In Kraichnan (1965) this was handled by treating the solenoidal part $\mathbf{u}^S(\mathbf{x},t|s)$ and the compressive or longitudinal part $\mathbf{u}^C(\mathbf{x},t|s)$ of $\mathbf{u}(\mathbf{x},t|s)$ as symmetrically as possible by the device of introducing a fictitious compressive part of $\mathbf{u}(\mathbf{x},t)$ that did not convect. For the present application, where we have no interest in $\mathbf{u}^C$, we apply solenoidal projection operators to both sides of (3.20), thereby obtaining the series for U_{in}^{SS}, the covariance of the solenoidal part only. Then we revert that series. The result is

$$\begin{aligned} U_{in}^0(\mathbf{x},t|s_1;\mathbf{x}',t|s_2) = {} & U_{in}^{SS}(\mathbf{x},t|s_1;\mathbf{x}',t|s_2) \\ & + \Big\{ -R_4 \int_{s_1}^t ds' \int_{s_2}^t ds''\, U_{jm}^{SS}(\mathbf{x},t|s';\mathbf{x}',t|s'')\,\nabla_j \nabla'_m U_{in}^{SS}(\mathbf{x},t|s_1;\mathbf{x}',t|s_2) \\ & - R_4 \int_{s_1}^t ds' \int_{s_1}^{s'} ds''\, U_{mj}^{SS}(\mathbf{x},t|s';\mathbf{x},t|s'')\,\nabla_j \nabla_m U_{in}^{SS}(\mathbf{x},t|s_1;\mathbf{x}',t|s_2) \\ & - R_4 \int_{s_2}^t ds' \int_{s_2}^{s'} ds''\, U_{mj}^{SS}(\mathbf{x}',t|s';\mathbf{x}',t|s'')\,\nabla'_j \nabla'_m U_{in}^{SS}(\mathbf{x},t|s_1;\mathbf{x}',t|s_2) \\ & + \text{higher-order terms} \Big\}_S, \end{aligned} \quad (3.21)$$

where $\{\ \}_S$ denotes the part of the series solenoidal in both i and n.

Since we have taken the frozen-**u** case, each U in (3.16) may be written as U^0. To obtain the full LH expansion we now change all labelling times in these U^0 factors to t, substitute the series (3.21) for each U^0 so altered and collect terms. The result is

$$\partial G(\mathbf{x},t)/\partial t = \sum_{n=1}^{\infty} K_{2n}(t)\nabla^{2n}G(\mathbf{x},t), \quad G(\mathbf{x},0) = \delta(\mathbf{x}), \tag{3.22}$$

where

$$K_2(t) = \frac{1}{3}\int_0^t U^L(s)\,ds + \text{nothing more}, \tag{3.23}$$

$$K_4(t) = \tfrac{1}{9}(R_4-1)\int_0^t ds\int_0^t ds'\int_0^{s'} ds''\, U^L(t-s)\,U^L(t|s';t|s'') + \text{higher-order terms} \tag{3.24}$$

and

$$U^L(t-s) = U_{ii}^{SS}(\mathbf{x},t|t;\mathbf{x},t|s), \quad U^L(t|s;t|s') = U_{ii}^{SS}(\mathbf{x},t|s;\mathbf{x},t|s'). \tag{3.25}$$

Again the leading term of each $K_{2n}(t)$ is of degree n in U functions. The $K_{2n}(t)$ are the same functions as the $K_{2n}^0(t)$ in (3.15), but they are expressed as different infinite series and we use different symbols to emphasize this.

The most striking difference from the bastard expansion (3.16) is that (3.23) contains only one term, Taylor's (1921) exact expression for the eddy diffusivity.† If the **u** distribution is Gaussian ($R_n = 1$), the leading terms of each $K_{2n}(t)$, $n \geqslant 2$, vanish, but the higher-order terms do not. Thus even in the Gaussian case there are non-vanishing coefficients of the higher derivatives of G in (3.22). The higher-derivative terms cause $G(\mathbf{x},t)$ to depart from a Gaussian shape and, since the leading term of (3.24) vanishes, these corrections to the Gaussian form first appear at the sixth order of perturbation theory and are proportional to t^6 at small t. The corrections to the leading term of (3.23) all disappear, whether or not **u** is Gaussian, because the higher terms in $K_2^0(t)$ are all precisely cancelled by the higher terms in (3.21), on substitution.

The leading term of each $K_{2n}(t)$ does not involve space derivatives of the covariance. For the degenerate case of a spatially uniform **u**, $U(t)$ and $U^L(t)$ are the same, the series $K_{2n}^0(t)$ and $K_{2n}(t)$ are identical, and each reduces to its leading term. This case is equivalent to the random linear oscillator of §2 but with, in general, a time-dependent parameter $a(t)$. The higher terms of each $K_{2n}(t)$ all involve spatial derivatives of the covariance.

The moments $\langle x_1^{2n}\rangle = \int G(\mathbf{x},t)\,x_1^{2n}\,dx$, where x_1 is any vector component of $\mathbf{x}$, are fixed according to (3.32) by the $K_{2m}(t)$ for $m \leqslant n$. Thus

$$\langle x_1^2\rangle = 2\int_0^t K_2(s)\,ds, \quad \langle x_1^4\rangle = 3\langle x_1^2\rangle^2 + 4!\int_0^t K_4(s)\,ds, \ldots. \tag{3.26}$$

This shows explicitly how non-vanishing values of the $K_{2n}(t)$ for $n > 1$ imply departures of $G(\mathbf{x},t)$ from a Gaussian shape. If **u** is Gaussian, the leading non-zero contribution to $K_4(t)$ at small t is $O(t^6)$.

Departure of $G(\mathbf{x},t)$ from a Gaussian form at intermediate t has been noted in computer simulations with a frozen, Gaussian **u** (Kraichnan 1970*b*). There appears to be a simple explanation. At times such that a typical fluid element has wandered the order of a correlation length, fluid elements with higher initial velocities are more likely to have gone far enough to suffer large deflexions. The result is a suppression of

† A detailed discussion is given in §7 of Kraichnan (1965).

the skirts of $G(\mathbf{x},t)$, which is consistent with the decreased value of $\langle x_1^4\rangle/\langle x_1^2\rangle^2$ noted in the simulations. At times long enough that typical fluid elements have travelled many correlation lengths the dispersion is effectively a random walk, and $G(\mathbf{x},t)$ should approach Gaussian form again as $t\to\infty$. Since $K_2(t)$ approaches a constant value as $t\to\infty$, (3.26) then implies that the exact $K_4(t)$ grows more slowly than t at large t.

If $\mathbf{u}$ is homogeneously distributed, the RGT (3.1), with a Gaussian, isotropic $\mathbf{v}$, transforms the generalized velocity covariance according to

$$U_{ij}(\mathbf{x},t|s;\mathbf{x}',t'|s') = \exp\left[\tfrac{1}{2}v_0^2(t-t')^2\,\nabla^2\right][U_{ij}(\mathbf{x},t|s;\mathbf{x}',t'|s')]_0, \tag{3.27}$$

of which (3.7) is a special case. Here U_{ij} is the covariance of the fluctuating velocity only; the total velocity field is $\mathbf{v}+\mathbf{u}$, with covariance $\delta_{ij}v_0^2+U_{ij}$. Equation (3.27) shows that the purely Lagrangian functions $U_{ij}(\mathbf{x},t|s;\mathbf{x}',t|s')$ and, in particular, $U^L(t)$ are invariant.

Under the RGT, $G(\mathbf{x},t)$ transforms according to (3.6) and the formally exact equation (3.22) is augmented by the term $v_0^2\nabla^2G(\mathbf{x},t)$ on the right-hand side. With this term added the $K_{2n}(t)$ are invariant under the RGT. The point of prime interest is that the LH expansions (3.23) and (3.24) and the corresponding expansions for the higher $K_{2n}(t)$ are invariant order by order. In order to assert this it is not enough to observe that the expansions are built from pure Lagrangian functions, invariant under (3.27). Instead, it must be shown that if the entire analysis is repeated with $\mathbf{u}(\mathbf{x})$ replaced by $\mathbf{v}+\mathbf{u}(\mathbf{x}-\mathbf{v}t)$ then, with the $v_0^2\nabla^2G(\mathbf{x},t)$ term added to (3.22), v_0^2 drops completely out of the expansions for the $K_{2n}(t)$. For $K_2(t)$, which consists of a single term, this is immediately assured by the invariance of $U^L(t)$. For $n\geqslant 2$, the asserted invariance property can be inferred as follows. The final expansions will consist of the terms found without $\mathbf{v}$, which are invariant, and, possibly, terms involving v_0^2 only and cross-terms involving both v_0^2 and U_{ij}. The terms in v_0^2 alone vanish order by order because $\mathbf{v}$ is Gaussian and for a Gaussian uniform field all the $K_{2n}(t)$ but $K_2(t)$ vanish identically. For the possible cross-terms, we note that the summed series $K_{2n}(t)$ is invariant, otherwise (3.6) would be violated, and the coefficients of the v_0^2 products are invariant U_{ij} functions. Since the latter are essentially arbitrary, according to the choice of $\mathbf{u}$, and v_0^2 is also arbitrary, there must be cancellation individually for each power of v_0^2 and each functional power of U_{ij}, which can happen only if the cross-terms cancel at each order separately.

The most striking difference between (3.3) and (3.22), apart from the fact that (3.4) is not invariant order by order to an RGT while (3.23) and (3.24) are, is that (3.3) is non-local in space and time while (3.22) is a pure differential equation for $G(\mathbf{x},t)$. Also, the $K_{2n}(t)$ in (3.22) are independent of G, while $\eta(\mathbf{x},t)$ is a functional power series in G. The non-localness in (3.3) is physically plausible because the dispersion process has an effective mean free path of a correlation length of $\mathbf{u}$ and an effective time between collisions of an eddy circulation time of $\mathbf{u}$ and we should not expect *a priori* that a local differential equation could describe the process exactly. Nevertheless, in the degenerate case of a uniform, Gaussian field (infinite mean free path), the lowest-order truncation of (3.22) is exact while the lowest-order truncation of (3.4) used in (3.3) gives an unphysical cusp-like behaviour, as mentioned earlier.

Actually, non-localness in space and time is implicit in (3.22)–(3.24) because the one-point Lagrangian velocity covariances therein are non-local expressions in the

Eulerian field. Moreover, the infinite series in ∇^2 in (3.22) is equivalent to an integral operator. That is, if we knew how to construct $f(\mathbf{x},t)$ we could write (3.22) in the form

$$\partial G(\mathbf{x},t)/\partial t = \int f(\mathbf{x}-\mathbf{y},t)\nabla^2 G(\mathbf{y},t)\,dy, \tag{3.28}$$

where in the case of a spatially uniform $\mathbf{u}$ field $f(\mathbf{x},t)$ degenerates to

$$f(\mathbf{x},t) = \delta(\mathbf{x})\,K_2(t) \quad (\text{uniform } \mathbf{u}). \tag{3.29}$$

The integral form (3.28) is to be preferred over (3.22). Suppose that the $K_{2n}(t)$ and hence the moments of the distribution $G(\mathbf{x},t)$ are known for $n \leqslant N$ and that (3.22) is then truncated at $n = N$. For $N > 1$, literal solution of the truncated equation may give unphysical oscillations in G and singularities. It would be better instead to use the known moments in some appropriate orthogonal expansion scheme (for example, see Kraichnan 1970*c*) to construct a smooth approximation to G. The approximation scheme could then, if desired, be translated into approximations to $f(\mathbf{x},t)$. Taken as they are, (3.22)–(3.24), plus the equations for the higher $K_{2n}(t)$, should be considered an expansion for the moments of $G(\mathbf{x},t)$, as expressed by (3.26), rather than an expansion for G itself.

Since the $K_{2n}(t)$ are expressed in terms of the Lagrangian covariance while the prescribed data are the Eulerian covariance of the frozen field $U_{ij}(\mathbf{x}-\mathbf{x}')$, the LH expansion for $G(\mathbf{x},t)$ is not logically complete without an LH renormalized expansion for the Lagrangian covariance. We show how to do this now, carrying the results explicitly only to the lowest terms. By (3.11),

$$\partial U_{in}(\mathbf{x},t|t;\mathbf{x}',t|s')/\partial t = S_{ni}(\mathbf{x}',t|s';\mathbf{x},t|t), \tag{3.30}$$

$$\partial U_{in}(\mathbf{x},t|s;\mathbf{x}',t|s')/\partial t = S_{in}(\mathbf{x},t|s;\mathbf{x}',t|s')+S_{ni}(\mathbf{x}',t|s';\mathbf{x},t|s), \tag{3.31}$$

where
$$S_{in}(\mathbf{x},t|s;\mathbf{x}',t|s') = \langle u_j(\mathbf{x})\,[\partial u_i(\mathbf{x},t|s)/\partial x_j]\,u_n(\mathbf{x}',t|s')\rangle \tag{3.32}$$

and we have noted that $u_i(\mathbf{x},t|t) = u_i(\mathbf{x})$. The primitive expansion for S_{in} is constructed by substituting (3.19) for each factor on the right-hand side of (3.32) and averaging explicitly. Then the labelling times are all changed to t, and each U^0 factor is expressed in U factors by (3.21). The result is

$$\begin{aligned} S_{in}(\mathbf{x},t|s;\mathbf{x}',t|s') = {}& R_4\int_s^t ds''\,U^{SS}_{jm}(\mathbf{x},t|t;\mathbf{x},t|s'')\,\partial^2 U^{SS}_{in}(\mathbf{x},t|s;\mathbf{x}',t|s')/\partial x_j\partial x_m \\ &+R_4\int_{s'}^t ds''\,U^{SS}_{jm}(\mathbf{x},t|t;\mathbf{x}',t|s'')\,\partial^2 U^{SS}_{in}(\mathbf{x},t|s;\mathbf{x}',t|s')/\partial x_j\partial x'_m \\ &+\text{higher-order terms}, \end{aligned} \tag{3.33}$$

where terms with single space derivatives of the covariance, which vanish because of the mirror symmetry, have been omitted. Since U^{SS}_{ij} is just the doubly solenoidal part of U_{ij}, (3.30), (3.31) and (3.33) form a closed set which evolves U_{in} from the boundary values

$$U_{in}(\mathbf{x},s|s;\mathbf{x}',s|s) = U_{in}(\mathbf{x}-\mathbf{x}'). \tag{3.34}$$

The LHDI approximation for U_{in} is obtained by retaining in (3.33) only the terms shown explicitly. The LHDI equations have the closed subset

$$\partial U^L(\mathbf{x},t)/\partial t = \tfrac{1}{3}R_4\int_0^t ds\,U^L(s)\nabla^2 U^L(\mathbf{x},t), \tag{3.35}$$

where $$U^L(\mathbf{x}-\mathbf{x}', t-s) = U_{ii}^{SS}(\mathbf{x}, t|t; \mathbf{x}', t|s), \quad U^L(t) = U^L(0, t). \tag{3.36}$$

If the frozen field is confined to a thin shell in k space, so that

$$U_{ii}(\mathbf{x}) = 3u_0^2 \sin(k_0 x)/k_0 x, \tag{3.37}$$

where u_0^2 is the mean-square velocity in any direction and k_0 is the shell wavenumber, then $\nabla^2 U^L(\mathbf{x}, t) = -k_0^2 U^L(\mathbf{x}, t)$ and (3.35) has the analytic solution

$$U^L(t) = 3u_0^2 \operatorname{sech}^2[(\tfrac{1}{2}R_4)^{\frac{1}{2}} u_0 k_0 t], \quad U^L(\mathbf{x}, t) = U^L(t) \sin(k_0 x)/k_0 x. \tag{3.38}$$

The re-expression of the $K_{2n}^0(t)$ of (3.16) in terms of Lagrangian covariances and the subsequent renormalization of the series for S_{in} can also be carried out with the ALH prescription explained in §§1 and 2, since the $K_{2n}(t)$ involve only pure Lagrangian functions. There is no change in (3.23), but (3.24) changes to

$$K_4(t) = \tfrac{1}{9}(R_4 - 1) \int_0^t ds \int_0^t ds' \int_0^{s'} ds''\, U^L(t-s)\, U^L(s'-s'') + \text{higher-order terms} \tag{3.39}$$

and all the higher $K_{2n}(t)$ have altered forms. The closed set for U_{ij} is (3.33) with $s' = t$ and appropriate labelling changes in the higher terms not shown explicitly, together with (3.30). The ALHDI approximation, which retains only the terms shown explicitly, gives just (3.35) again.

Both the LH and the ALH treatment may be applied also to the case of time-dependent random convecting fields $\mathbf{u}(\mathbf{x}, t)$ by adopting (3.12) and paying the price of substantial additional complication.

4. Consistency properties of the expansions

The exact Navier–Stokes and passive-scalar dynamics exhibit some basic invariance and conservation properties. We have already discussed random Galilean invariance. The Navier–Stokes equation conserves energy, except for viscous dissipation, and similarly the passive-scalar convection conserves scalar variance. There is an additional conservation property of the action of the labelling transformation on the generalized fields (Kraichnan 1965), examples of which are

$$\frac{\partial}{\partial t}\int u_i(\mathbf{x}, t|s)\, u_j(\mathbf{x}, t|s')\, d\mathbf{x} = 0, \quad \frac{\partial}{\partial t}\int \psi(\mathbf{x}, t|s)\, \psi(\mathbf{x}, t|s')\, d\mathbf{x} = 0. \tag{4.1}$$

All of the quadratic conservation properties for averages arise from the identical vanishing of space integrals of triple moments with simultaneous labelling times. Moreover, these integrals vanish identically, whatever the values of the zero-superscripted functions, at each order of the primitive expansions for the triple moments. This can be inferred by putting an ordering parameter λ in front of the stochastically nonlinear terms in the equations of motion and noting that the primitive expansion is a power series in λ, which can vanish for arbitrary λ only if the coefficient of each power vanishes. Or it can be noted that the triple-moment integral vanishes identically whatever the values of the essentially arbitrary zero-superscripted functions and therefore each functional power of these functions in the primitive expansion of the integral must vanish individually.

The conservation properties survive also at each order of the Eulerian, LH and ALH renormalized expansions for the triple moment. This follows directly from the fact that the renormalized expansions are obtained by a one-to-one reversion of the relation between zero-subscripted and exact functions, so that identical vanishing as a functional of the former implies identical vanishing as a functional of the latter. In the Eulerian renormalization, the one-to-one transformation is between the functions with the same arguments. In the LH reversion, the transformation is between, for example, $U^0_{ij}(\mathbf{x},t|s;\mathbf{x}',t|s')$ and $U^{SS}_{ij}(\mathbf{x},t|s;\mathbf{x}',t|s')$, where t is a fixed parameter during the reversion. Note that only simultaneous labelling times enter the triple moments involved in the conservation laws, so that the LH prescription gives only a single labelling time in the final series. In the ALH reversion, the transformation is between, for example, $U^0_{ij}(\mathbf{x},t|s;\mathbf{x}',t|s')$ and $U^{SS}_{ij}(\mathbf{x},s|s;\mathbf{x}',s|s')$ $(s \geqslant s')$. The one-to-one character of this transformation follows from the fact that

$$U^0_{ij}(\mathbf{x},t|s;\mathbf{x}',t|s') = U^0_{ij}(\mathbf{x},s|s;\mathbf{x}',s|s')$$

together with the symmetry of U^0_{ij} to exchange of its two argument sets.

All the named conservation properties have been verified in detail for the lowest truncations of all the expansions (Kraichnan 1965).

A further property of the exact dynamics is the inviscid fluctuation–dissipation relation in absolute equilibrium:

$$U_{ij}(\mathbf{x},t|s;\mathbf{x}',t'|s') = G_{ij}(\mathbf{x},t|s;\mathbf{x}',t'|s') \quad (s \geqslant s'), \tag{4.2}$$

which holds, with proper normalization of U_{ij}, for a system cut off at some high wavenumber (Kraichnan 1965). This relation holds for the primitive expansion in the sense that the expressions for t and t' derivatives of U_{ij} and G_{ij} are identical at each order of expansion if (4.2) is used in these expansions. The same kind of argument as for the conservation properties then shows that (4.2) survives at each order of the Eulerian renormalized expansion. In the case of the LH and ALH renormalized expansions, the argument is valid if $t = t'$ (the ALH expansion is undefined for $t \neq t'$). But it can not be asserted from this argument that (4.2) survives in the LH renormalized series order by order if $t \neq t'$. This is because both t and t' appear as labelling times before reversion and a given zero-superscripted function may end up being expressed in a series with only t labels, one with only t' labels, or one with both t and t' labels, depending on how the function arises in the primitive perturbation expansion; the reversion transformation is formally exact if $t \neq t'$ but it is not one-to-one.

In connexion with the LHDI approximation, the first truncation of the LH expansion, it was pointed out (Kraichnan 1965) that (4.2) is recovered for $t \neq t'$ if the relabelling prescription for reversion of triple moments describing differentiation of U_{ij} or G_{ij} with respect to t is modified. Doing this destroys the symmetry between (2.29) and (2.38), however, and we do not adopt here the possible modification discussed in Kraichnan (1965, § 11).

Since the final Eulerian and LH renormalized expansions are for the derivatives of $U_{ij}(\mathbf{x},t|s;\mathbf{x}',t'|s')$ with respect to t and t', an additional consistency question is whether these expansions satisfy integrability conditions in the t, t' plane, order by order. This question can be attacked by methods like that used above for the conservation properties, but there are some delicate points and we defer discussion to another place. It can be shown by other methods that the Eulerian renormalized expansion does satisfy integrability order by order. Since the LH expansion is exact at lowest order for the random-oscillator problem it satisfies integrability order by order in that case at least. The question does not arise for the closed subset of the LH equations which describes the purely Lagrangian ($t = t'$) functions or for the ALH expansion.

Arguments based on model systems suggest that in general the primitive expansions for the Navier–Stokes and passive-scalar problems diverge for all t but that finite radii of convergence in t can be obtained by taking an initial statistical distribution that bounds initial amplitudes and using a high wavenumber cut-off (Kraichnan 1966, 1970c). Gaussian initial statistics and an infinite wavenumber cut-off can then be approached as a limit. Even in the case of the Gaussian random oscillator of §2, where the expansion of $G(t)$ in powers of t is absolutely convergent for all t, truncations do not give uniform approximations that permit approximation of quantities like

$$\int_0^\infty G(t)\,dt.$$

The reverted expansions of zero-superscripted functions as functional series in exact functions must diverge if the primitive expansions do. But it is not easy to say what happens to analyticity properties when the renormalized expansions are formed by substituting these divergent series into the divergent primitive expansions for triple moments and time derivatives of Green's functions. In general it must be assumed that the renormalized expansions diverge also and that convergent approximations must be formed from these expansions by more powerful procedures than truncation. Even in the case of the Gaussian random oscillator, truncations of the Eulerian renormalized expansion for $\langle a\hat{G}(t)\rangle$ give solutions of the integro-differential equation for $G(t)$ which blow up (Kraichnan 1961).

The lowest truncation of the Eulerian renormalized series (the direct-interaction approximation) is exceptional in that it describes exactly a stochastic model system and thereby assures healthy solutions (Kraichnan 1961). The LHDI and ALHDI approximations do not give this assurance. But they do keep the conservation laws and can be obtained from the direct-interaction approximation by an updating of labelling times, which should increase the stability of the equations. The consistency of the LHDI and ALHDI approximations is supported by the existing experience with them.

It is highly interesting that the LH expansion is exact at lowest order for the Gaussian random oscillator. A more elaborate LH expansion, involving vertex renormalization, is in addition exact at lowest order when a is exponentially distributed, so that the exact Green's function is $G(t) = (1-i\langle a\rangle t)^{-1}$, which has a finite rather than an infinite radius of convergence. These examples serve to show that the Eulerian and LH renormalizations can have importantly different convergence properties. We shall not describe the vertex renormalization here.

5. Concluding remarks

The LH and ALH renormalized expansions presented and illustrated here provide a logical and systematic basis for the LHDI and ALHDI approximations. The latter appear in their own right instead of resulting from heuristic modifications of the Eulerian direct-interaction approximation.

We have mentioned that vertex renormalization, in which certain relations are written in terms of effective interaction coefficients which obey integral equations, can be carried over to the LH expansions. The combination of series reversion and labelling changes which we have used to obtain the LH expansions given above is in fact highly flexible and can be used also to generate expansions which have no meaningful description in terms of the diagram methods of traditional field theory. In addition, the expansions as presented here can easily be generalized to include random driving forces and can also be enlarged to include non-zero initial values of triple moments.

The key ingredient in the LH expansions is the labelling transformations (1.1) and (2.2), which introduce a redundant description of the basic fields. It has seemed most logical to apply these transformations to $\mathbf{u}$ and ψ, as we have done, and this is supported by the nice form found for the LH expansion of the passive-scalar Green's function. However we could also have taken the vorticity or rate-of-strain field instead of the velocity field and the scalar-gradient field instead of the scalar field as basic. In the Eulerian treatment, it makes no difference. But different LH expansions do result from applying the labelling transformation to the derivative fields instead of to $\mathbf{u}$ and ψ themselves. Thus if (1.1) were replaced by

$$\partial\boldsymbol{\omega}(\mathbf{x},t|s)/\partial t = \mathbf{u}(\mathbf{x},t).\nabla\boldsymbol{\omega}(\mathbf{x},t|s), \quad \boldsymbol{\omega}(\mathbf{x},s|s) = \boldsymbol{\omega}(\mathbf{x},s), \tag{5.1}$$

where $\boldsymbol{\omega}(\mathbf{x},t)$ is the Eulerian vorticity field, a very different expansion would result. This is most easily seen by considering the two-dimensional inviscid case, in which (5.1) is identical with the equation obeyed by $\boldsymbol{\omega}(\mathbf{x},t)$. There is then no dependence on measuring times for $\boldsymbol{\omega}(\mathbf{x},t|s)$, and in the final LH expansion the Lagrangian moments of $\boldsymbol{\omega}$ which enter would have infinite correlation times. A more physical choice would appear to be making the rate-of-strain field basic so that the memory times in the resulting LHDI approximation for triple moments would relate directly to the straining process.

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Appendix. Reversion of functional power series

Let $f(x)$ be given by

$$f(x) = g(x) + \sum K_2(x,y_1,y_2)\,g(y_1)\,g(y_2) + \sum K_3(x,y_1,y_2,y_3)\,g(y_1)\,g(y_2)\,g(y_3) + \dots, \tag{A 1}$$

where x may include both continuous arguments, such as space-time position, and discrete arguments, such as vector indices, and $\sum$ denotes summation and integration over all the intermediate arguments y_n. We revert (A 1) by successive approximation, starting with the zeroth approximation $g(x) = f(x)$. Substituting this approximation for $g(x)$ into all the $\sum$'s in (A 1) and retaining all terms up to the second degree in f we obtain the first approximation

$$g(x) = f(x) - \sum K_2(x,y_1,y_2)f(y_1)f(y_2). \tag{A 2}$$

Substituting (A 2) for g in all the $\sum$'s in (A 1) and retaining all terms up to the third degree in f we obtain the second approximation

$$g(x) = f(x) - \sum K_2(x, y_1, y_2) f(y_1) f(y_2) - \sum K_3(x, y_1, y_2, y_3) f(y_1) f(y_2) f(y_3) + 2 \sum K_2(x, y_1, y_2) K_2(y_2, y_3, y_4) f(y_1) f(y_3) f(y_4), \quad \text{(A 3)}$$

where we have assumed, without loss of generality, that the K_n in (A 1) are fully symmetric in all the y arguments.

Repetition of this process gives $g(x)$ as a functional power series in f to any desired order.

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